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THE SUM OF NIL ONE-SIDED IDEALS OF BOUNDED INDEX OF A RING

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ABSTRACT

The sum $N(R)$ of nil one-sided ideals of bounded index of a ring R is shown to coincide with the set of all strongly nilpotent elements of R of bounded index. The known result that $N(R)$ is contained in the prime radical is highly improved and it is shown $N(R)$ is contained in $N_2(R)$. It is proved that the sum of a finite number of nil left ideals of bounded index has bounded index.

Introduction

A result of Amitsur [R, Th.2.6.27] shows that the set $N(R)$ of all elements of a ring R that generate one-sided ideals which are nil of bounded index is an ideal, so $N(R)$ is also the sum of all nil one-sided ideals of R of bounded index. The ideal $N(R)$ turns out to be of crucial significance in the problem of embedding PI-rings in matrix rings over commutative rings [R, Th. 6.1.26]. It is proved in [R, Prop.2.6.26] that $N(R)$ is contained in the prime radical of R. The prime radical of a ring has been characterized as the set of all strongly nilpotent elements of the ring [L, p. 56]. For a strongly nilpotent element one may define an index which might be infinite. We characterize $N(R)$ as the set of those elements of R which are strongly nilpotent of bounded index

The definition of the prime radical in terms of the nondecreasing transfinite sequence of ideals $N_\alpha(R)$ [R, I, p. 204] raises the question about the location of $N(R)$ relative to that sequence. It is clear that $N_1(R)$ — the sum of all nilpotent ideals of R — is contained in $N(R)$ and we prove that $N(R)$ is contained in

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$N_2(R)$ — the sum of all ideals of R which are nilpotent mod $N_1(R)$. Moreover, examples are given showing that each of the following three cases may occur: $N_1(R) = N(R) \neq N_2(R)$, $N_1(R) \neq N(R) = N_2(R)$, $N_1(R) \neq N(R) \neq N_2(R)$.

We conclude the paper with a new proof showing that $N(R)$ is an ideal. Our proof yields that the sum of a finite number of nil left ideals of bounded index has bounded index, a result which doesn't seem to follow from the known proof which shows $N(R)$ is an ideal.

1. Definitions and notations

Given an element a of a ring R let $(a)_\ell$ (resp. $(a)_r$) denote the left (right) ideal of R generated by a. We do not assume that R has a unit so we cannot write Ra instead of $(a)_\ell$ as in [R] and therefore we have

$$N(R) = \{a \in R \mid (a)_\ell \quad \text{is nil of bounded index}\}.$$

The definition of $N(R)$ is left-right symmetric since $(a)_\ell$ is nil of bounded index if and only if $(a)_r$ is nil of bounded index.

An element a of R is called **strongly nilpotent** provided every m-sequence starting with a is ultimately zero. Recall that an **m-sequence** is a sequence $a_0, a_1, a_2, \ldots$ of elements of R such that $a_n \in a_{n-1}Ra_{n-1}$, $n = 1, 2, \ldots$ Following [BR, p. 45] we prefer to call the elements of an m-sequence **quasipowers** of a_0. If $a_0 = a$ is given, a_n is said to be a quasipower of a of **order** n and will be denoted by $a^{(n)}$. Note that $a^{(n)}$ depends on n elements $b_1, \ldots, b_n$ of R in the following manner

$$a^{(0)} = a;$$

$$a^{(n)}(b_1, \ldots, b_n) = a^{(n-1)}(b_1, \ldots, b_{n-1}) b_n a^{(n-1)}(b_1, \ldots, b_{n-1}), \quad n \geq 1 .$$

It is clear that $a^{(u+v)} = (a^{(u)})^{(v)}$ for any two integers u, $v \geq 0$. Note also that

$$(xa)^{(n)}(b_1, \ldots, b_n) = xa^{(n)}(b_1 x, \ldots, b_n x).$$

Given a strongly nilpotent element a we say that it has **bounded index** if there exists an integer $n \geq 0$ such that all quasipowers of a of order n are 0. The least such n will be denoted by $\sigma(a)$ and if no such n exists we set $\sigma(a) = \infty$. Note that if a belongs to a nilpotent ideal of index k then $\sigma(a) \leq \lceil \log_2 k \rceil$ where for a real number x we denote by $\lceil x \rceil$ the smallest integer $\geq x$. We shall also use the notation $\lfloor x \rfloor$ for the largest integer $\leq x$.

2. A characterization of $N(R)$

Let F be the free ring without unit generated freely by $x_1, x_2, \dots$ and let I_n be the ideal of F generated by all n-th powers of elements of F. Then F/I_n is the countable generated generic nil ring of index n. By [A, Th.1], $(F/I_n)^m \subseteq N_1(F/I_n)$ where $m = \lfloor n/2 \rfloor$. In particular $x_1 \cdots x_m + I_n$ generates a nilpotent ideal of F/I_n and we denote the index of nilpotence of this ideal by $\rho(n)$. The function $\rho(n)$ is not known and only bounds for $\rho(n)$ are known when $n \le 7$.

Lemma 1: *For any nil ring S of index $\le n$ and any m elements $a_1, \dots, a_m$ of S, $m = \lfloor n/2 \rfloor$, the ideal generated by the product $a_1 \cdots a_m$ is nilpotent of index $\le \rho(n)$.*

Proof: We have to show that if $b_1, \dots, b_{\rho(n)-1}$ are arbitrary given elements of S then

$$(a_1 \cdots a_m) b_1 (a_1 \cdots a_m) \cdots b_{\rho(n)-1} (a_1 \cdots a_m) = 0.$$

Consider an homomorphism φ: $F \to S$ with $\varphi(x_i) = a_i$, $i = 1, \dots, m$, and $\varphi(x_{m+j}) = b_j$, $j = 1, \dots, \rho(n)-1$. Since S is nil of index n we have $I_n \subseteq \ker \varphi$. By the above-mentioned result for $x_1 \cdots x_m + I_n$ we get

$$(x_1 \cdots x_m) x_{m+1} (x_1 \cdots x_m) \cdots x_{m+\rho(n)-1} (x_1 \cdots x_m) \in I_n$$

and applying φ the desired result follows. ∎

Theorem 1: $N(R) = \{a \in R \mid \sigma(a) < \infty\}$.

Proof: Let $\sigma(a) = n < \infty$. If $n = 0$ then $a = 0 \in N(R)$. If $n \ge 1$ we have $a^{(n)}(b_1, \dots, b_n) = 0$ for any $b_1, \dots, b_n \in R$ and in particular $a^{(n)}(x, \dots, x) = 0$ for any $x \in R$ implying $(a)_\ell$ is nil of bounded index ($\le 2^n$ if $1 \in R$ and $\le 2^{n+1} - 1$ otherwise).

Now suppose $(a)_\ell$ is nil of index n. If $n = 1$ then $a = 0$ and $\sigma(a) = 0$. For $n \ge 2$ let $m = \lfloor n/2 \rfloor$ and let $u = \lceil \log_2(m+1) \rceil$. Then $2^u \ge m+1$ and any quasipower $a^{(u)}$ can be written as a product of $m+1$ elements $a_0, a_1, \dots, a_m$ belonging to $(a)_\ell$. Applying Lemma 1 to $S = (a)_\ell$ we get that the ideal of S generated by $a_1 \cdots a_m$ is nilpotent of index $\le \rho(n)$. So if $v = \lceil \log_2 \rho(n) \rceil$ then $2^v \ge \rho(n)$ and therefore $(a_1 \cdots a_m)^{(v)}(c_1, \dots, c_v) = 0$ for any $c_1, \dots, c_v \in S$. Now given any v elements $b_1, \dots, b_v \in R$ we have

$$(a_0 a_1 \cdots a_m)^{(v)}(b_1, \dots, b_v) = a_0 (a_1 \cdots a_m)^{(v)}(b_1 a_0, \dots, b_v a_0)$$

and

$$(a_1 \cdots a_m)^{(v)}(b_1a_0, \ldots, b_va_0) = 0$$

since $b_1a_0, \ldots, b_va_0 \in S$. But $a_0a_1 \cdots a_m = a^{(u)}$ so

$$a^{(u+v)} = (a^{(u)})^{(v)} = (a_0a_1 \cdots a_m)^{(v)} = 0$$

and therefore $\sigma(a) \leq u + v$. ∎

3. The location of $N(R)$ relative to $N_\alpha(R)$

For an algebra over a field of zero characteristic each $(a)_\ell$, $a \in N(R)$, is nilpotent by the Nagata-Higman Theorem [J] so $N(R) = N_1(R)$. In general it is known that $N(R)$ is contained in the prime radical. This result is improved in the following

THEOREM 2: $N(R) \subseteq N_2(R)$.

Proof: Let L be a nil left ideal of index $\leq n$. By [A, Th.1] $L^m \subseteq N_1(L)$ where $m = \lfloor n/2 \rfloor$. If A is a nilpotent ideal of L then LA is a nilpotent left ideal of R so $LA \subseteq N_1(R)$. Since $N_1(L)$ is the sum of nilpotent ideals of L we get $LN_1(L) \subseteq N_1(R)$ so $L^{m+1} = LL^m \subseteq N_1(R)$. But $N_2(R)$ contains the left ideals of R which are nilpotent mod $N_1(R)$ so $L \subseteq N_2(R)$ and therefore $N(R) \subseteq N_2(R)$. ∎

COROLLARY 1: *Let L be a nil left ideal of R of bounded index n and $m = \lfloor n/2 \rfloor$. Then for any $a_1, \ldots, a_{m+1} \in L$ the product $a_1a_2 \cdots a_{m+1}$ generates a nilpotent ideal of R of index $\leq \rho(n)$.*

Proof: The claim about the nilpotency of the ideal generated by $a_1a_2 \cdots a_{m+1}$ follows since we have shown $L^{m+1} \subseteq N_1(R)$. The claim about the index follows from Lemma 1 since if $b_1, \ldots, b_k \in R$, where $k = \rho(n) - 1$, we have

$$\begin{aligned}&(a_1a_2 \cdots a_{m+1})b_1(a_1a_2 \cdots a_{m+1}) \cdots b_k(a_1a_2 \cdots a_{m+1}) \\ =&a_1(a_2 \cdots a_{m+1})(b_1a_1)(a_2 \cdots a_{m+1}) \cdots (b_ka_1)(a_2 \cdots a_{m+1}) = 0\end{aligned}$$

since $b_1a_1, \ldots, b_ka_1 \in L$. ∎

Under the assumptions of Corollary 1 we get in particular that if $a \in L$ then a^{m+1} generates a nilpotent ideal of R of index $\leq \rho(n)$. But in this case it is possible to get a concrete bound for the index. By [K, Th.6] if y is an element of the infinitely generated generic nil ring of index n then y^m generates a nilpotent ideal of L of index $\leq 2^{n-m}$. It follows that a^m generates a nilpotent ideal of L of index $\leq 2^{n-m}$. So by the proof of Corollary 1 we get

COROLLARY 2: *Let L be a nil left ideal of R of bounded index n and $m = \lfloor n/2 \rfloor$. If $a \in L$ then a^{m+1} generates a nilpotent ideal of R of index $\leq 2^{n-m}$.*

Now we know that $N_1(R) \subseteq N(R) \subseteq N_2(R)$ and we are going to show that all three possibilities may occur namely: (i) $N_1(R) = N(R) \neq N_2(R)$; (ii) $N_1(R) \neq N(R) = N_2(R)$; (iii) $N_1(R) \neq N(R) \neq N_2(R)$.

(i) R may be any PI-algebra over a field of zero characteristic with $N_2(R) \neq N_1(R)$. For instance let S be the quotient ring of the ring of polynomials $\mathbb{Q}[t_1, t_2, \ldots]$ modulo the ideal generated by $\{t_i^2 \mid i = 1, 2, \ldots\}$ and let

$$R = \begin{pmatrix} N_1(S) & S \\ N_1(S) & N_1(S) \end{pmatrix}.$$

Then $R = N_2(R) \neq N_1(R) = M_2(N_1(S))$.

(ii) We take S and R as in (i) with $\mathbb{Q}$ replaced by $\mathbb{Z}_2$. Here $N_1(S)$ satisfies the identity $x^2 = 0$ and R satisfies the identity $x^4 = 0$ so we have $R = N(R) = N_2(R)$ and $R \neq N_1(R)$ since e_{12} does not generate a nilpotent ideal. Indeed, the ideal generated by e_{12} contains the elements $e_{12}(t_i e_{21}) = t_i e_{11}$ and $t_1 \cdots t_n \neq 0$ for any $n \geq 1$.

Another example of type (ii) is F/I_4. Indeed, the ring R we have just constructed is a homomorphic image of F/I_4 and since $R \neq N_1(R)$ we have $F/I_4 \neq N_1(F/I_4)$. But $(F/I_4)^2 \subseteq N_1(F/I_4)$ so $F/I_4 = N_2(F/I_4)$.

(iii) Let $R = R_1 \oplus R_2$ where R_1 is a ring of type (i) and R_2 is a ring of type (ii). Then $N(R) = N(R_1) \oplus N(R_2) = N_1(R_1) \oplus N_2(R_2) \neq N_1(R), N_2(R)$ since $N_1(R_i) \neq N_2(R_i)$ for $i = 1, 2$.

4. A new proof showing $N(R)$ is an ideal

Considering the known proof showing $N(R)$ is an ideal one may wonder whether it is possible to prove this result inside R. We prove that the sum of two nil left ideals of bounded index has bounded index and this easily implies $N(R)$ is an ideal. Our result is interesting in its own right and doesn't seem to follow from [R, Th.2.6.27].

If $(a)_\ell$ is nil of bounded index then it is clear that $(xa)_\ell$ and $(ax)_\ell$ are nil of bounded index for any $x \in R$. Moreover, we have $\sigma(xa) \leq \sigma(a)$ since

$(xa)^{(n)}(b_1, \ldots, b_n) = xa^{(n)}(b_1x, \ldots, b_nx)$ and similarly $\sigma(ax) \leq \sigma(a)$. It remains to show that $N(R)$ is closed under addition. Theorem 3 will say more than that. The generic nil ring of index k generated by s elements is nilpotent and we denote its index of nilpotence by $\nu(k, s)$. Thus any nil ring of index k generated by s elements is nilpotent of index $\leq \nu(k, s)$ [K].

THEOREM 3: *Given two positive integers m, n there exists an integer $\tau(m, n)$ such that for any ring R and any two nil left ideals L_1, L_2 of R of indices m, n respectively $L_1 + L_2$ is nil of index $\leq \tau(m, n)$.*

Proof: Let a, b be any two given elements of L_1, L_2 respectively. We prove that there exists an integer $\tau(m, n)$ which does not depend on a, b such that $a + b$ is nilpotent of index $\leq \tau(m, n)$.

Consider the subring S of R generated by a, b and let A be the left ideal of S generated by a and B the left ideal of S generated by b. We have $S = A + B$ and $A \subseteq L_1$, $B \subseteq L_2$ so A is a nil ring of index m and B is a nil ring of index n. It is clear that A is generated as a ring by the finite set $\{b^j a^i \mid i = 1, \ldots, m-1;\ j = 0, \ldots, n-1\}$ and B is generated by $\{a^i b^j \mid i = 0, \ldots, m-1;\ j = 1, \ldots, n-1\}$. It follows that A is nilpotent of index $\leq \nu_1 = \nu(m, (m-1)n)$ and B is nilpotent of index $\leq \nu_2 = \nu(n, (n-1)m)$. It follows that the ring $S = A + B$ is nilpotent of index $\leq \nu_1 + \nu_2 - 1$ and therefore there exists a bound for the index of nilpotence of S depending only on m and n and such a bound may be defined as $\tau(m, n)$ since $a + b \in S$. ∎

COROLLARY 3: *In any ring the sum of a finite number of nil left (right) ideals of bounded index is a nil left (right) ideal of bounded index.*

References

[A] S. A. Amitsur, *Nil PI-rings,* Proc. Amer. Math. Soc. **2** (1951), 538–540.

[BR] J. Berstel and C. Reutenauer, *Rational Series and their Languages,* Springer-Verlag, 1988.

[J] N. Jacobson, *Structure of Rings,* Amer. Math. Soc. Colloq. Publ. 37, Providence, 1964.

[K] A. A. Klein, *Upper bounds for the indices of nilpotence of certain rings and ideals,* Comm. in Algebra **12** (1984), 447–466.

[L] J. Lambeck, *Lectures on Rings and Modules,* Blaisdell, Waltham, MA, 1966.

[R] L. H. Rowen, *Ring Theory I and II,* Academic Press, 1988.